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MONOTONICITY PARADOXES IN THE THEORY OF ELECTIONS

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An election procedure based on voter preference rankings is said to be monotonic if the alternative chosen by the procedure for any profile of voter preference rankings is also chosen after it is moved up in one or more of the profile's rankings. Several reasonable-sounding election procedures that are known to violate monotonicity are examined along with some new classes of nonmonotonic procedures. Closely-related procedures that are monotonic are also identified. The procedural mechanisms and combinatorial structures that give rise to failures of monotonicity are analyzed in some detail.

1. Introduction

Social choice theory [2, 3, 4, 5, 7, 14] has long been concerned with procedures that are designed to select a collectively best alternative or subset of alternatives from an available set on the basis of individual preference rankings of the alternatives. In this paper I shall examine a condition that has been proposed as an obviously compelling property that such procedures should satisfy, and demonstrate that it is violated by a number of apparently reasonable selection procedures, some of which are widely used in practice. These include certain runoff election methods and sequential majority elimination voting as well as a procedure that does not involve successive eliminations. It will be assumed throughout that the basic alternative set X is finite with at least three elements.

I shall refer to the condition in question as monotonicity, although closely related conditions are also known as nonnegative responsiveness and as positive association of social and individual values [1, 2]. The version used here says that if x is the unique collectively best alternative for a given profile π of individual preference rankings, and if profile π' is obtained from π by moving x up in some of the rankings, leaving all else unchanged, then x should be one of the collectively best alternatives for profile π'.

Monotonicity is clearly a desirable condition for social selection procedures, and I am not aware that it has been seriously challenged in the literature. Although

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violations of conditions related to monotonicity were recognized many years ago,[1] its formal introduction into the social choice literature (in a social ranking version) appears to be due to Arrow [1]. Later, Smith [15] observed that monotonicity fails for a class of social ranking procedures. Others [6, 9, 13] have noted violations of monotonicity that are similar to those discussed by Smith, with the exception of one procedure associated with the name of Dodgson (Lewis Carroll).

Later sections of the paper will examine the known monotonicity violators along with other classes of reasonable-sounding election procedures that also violate monotonicity. Since it is usually far from obvious that monotonicity fails for these procedures, and since various special cases and closely-related procedures do in fact satisfy monotonicity, I have used the phrase "monotonicity paradoxes" to describe the potentially unexpected failure of monotonicity. Once one sees why monotonicity can fail for various procedures, it may also come as a surprise to note that certain other procedures are monotonic. For example, a limiting case (denoted f_2) of a class of nonmonotonic procedures discussed in Section 4 turns out to be monotonic, and some generic procedures that are nonmonotonic for four or more alternatives are in fact monotonic when X has only three members.

A major aim of the paper is to get behind the bare statement of monotonicity and to understand the mechanisms of the procedures and their combinatorial structures that produce the counterexamples to monotonicity. For the most part, these structures are illustrated through the proofs of the theorems, which are therefore an integral part of the exposition. Two special binary relations for preference profiles are central to much of the analysis, namely a strong positional dominance relation and the strict simple majority relation. We discuss these further in the next section, which presents the formal background for our ensuing examination of specific election procedures.

2. Election procedures

A strict ranking of the set X of $m \geq 3$ alternatives will be written as $xy \cdots z$ when x is ranked first, y second, and z last. Alternative x is *ranked ahead of* y in a ranking if x precedes y in the ranking. Moreover, x *moves up* in going from ranking r to r' if all alternatives in $X \setminus \{x\}$ have the same order in r and r', and x is ranked ahead of more alternatives in r' than in r.

Let Π be the set of all nonempty finite lists or *profiles* of (strict) rankings on X. Each term (individual ranking) in a profile $\pi \in \Pi$ is viewed as a judge's or voter's ranking of the alternatives from best to worst. We shall often describe a profile by the number of voters who have each ranking on X. Thus, $\pi = (xyz, xyz, zyx)$ will be

[1] See, for example [12, p. 21] and [11, p. 93]. These passages were brought to my attention by Steven Brams, who was informed of them by Duff Spafford. I am indebted to Brams also for [6].

identified as

	π
xyz	2
zyx	1 .

This suggests that the election procedures considered below are anonymous (do not depend on the order in which the terms of π are listed), which is indeed true.

Let F be the set of all functions that map Π into the nonempty subsets of X. Each $f \in F$ is an *election procedure*, and each $f(\pi)$ is an *elected set.* Although many contexts require the selection of a single alternative, it is useful to allow the elected set to contain more than one alternative. We view $f(\pi)$ as the collectively best (socially most preferred) alternatives in X as determined by f acting on π. Accordingly, one might propose that every reasonable election procedure should satisfy

Monotonicity. *If $f(\pi) = \{x\}$, and if π' is obtained from π by moving x up in one or more terms of π, then $x \in f(\pi')$.*

Many election procedures are obviously monotonic, and others can be shown to be monotonic with a little effort. For example, if $|X| = 3$ and $1, \lambda_2$ and 0 points are assigned, respectively, to each voter's first, second, and third-ranked alternatives, and if $f(\pi)$ consists of the alternatives with the largest point total, then f is monotonic so long as $1 \geq \lambda_2 \geq 0$. Other monotonic examples are provided by many Condorcet social choice functions [9], although some of these are not monotonic, as will be noted below.

Before proceeding to our theorems and examples concerning nonmonotonic election procedures, we define some of the key notions used in the analysis. Two binary relations for each $\pi \in \Pi$ will be used several places. The first is $M(\pi)$, defined by: $xM(\pi)y$ if more terms of π have x ranked ahead of y than have y ranked ahead of x. This is Condorcet's strict majority relation, and we shall say that $f \in F$ is *Condorcet* if $f(\pi) = \{x\}$ whenever $xM(\pi)y$ for all $y \in X \setminus \{x\}$. Frequent use will be made of profiles for which $M(\pi)$ is nontransitive and in fact cyclic, as when $xM(\pi)yM(\pi)zM(\pi)x$ for $\pi = (xyz, zxy, yzx)$.

We shall also use "majority" counts and define $\hat{\pi}(x, y)$ as the number of terms of π in which x is ranked ahead of $y \neq x$. The corresponding "majority" proportion is $\pi(x, y)$, defined by

$$\pi(x, y) = \frac{\hat{\pi}(x, y)}{\hat{\pi}(x, y) + \hat{\pi}(y, x)},$$

so that $\pi(x, y) + \pi(y, x) = 1$ for distinct x and y.

The second relation alluded to above is an asymmetric and transitive strong positional dominance relation $D(\pi)$ that is based on position counts. Let $\pi(x_i)$ be the number of terms of π that rank x in position i, for $i=1,\dots,m$ when $|X|=m$. Clearly, $\pi(x_1)+\cdots+\pi(x_m)=\pi(y_1)+\cdots+\pi(y_m)$ for any $x, y\in X$. The relation $D(\pi)$ is defined by

$$xD(\pi)y \quad \text{iff} \quad \sum_{i=1}^{k} \pi(x_i) > \sum_{i=1}^{k} \pi(y_i) \quad \text{for } k=1,\dots,m-1,$$

when $|X|=m$.

Relation $D(\pi)$ will be used in connection with positional scoring. We shall let Λ_m be the set of all nonincreasing ($\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_m$) and nonconstant ($\lambda_1>\lambda_m$) m-dimensional real vectors $\lambda=(\lambda_1,\dots,\lambda_m)$, and refer to $\lambda\in\Lambda_m$ as a *score vector.* The following lemma connects D and λ.

Lemma 1. *If $|X|=m$, $\lambda\in\Lambda_m$ and $xD(\pi)y$, then $\sum_{i=1}^{m}\lambda_i\pi(x_i)>\sum_{i=1}^{m}\lambda_i\pi(y_i)$.*

Proof. Summation by parts gives

$$\sum_{i=1}^{m} \lambda_i[\pi(x_i)-\pi(y_i)] = \sum_{k=1}^{m-1}\left[\sum_{i=1}^{k}(\pi(x_i)-\pi(y_i))\right](\lambda_k-\lambda_{k+1}),$$

and the lemma's hypotheses imply that the right hand side is positive. □

We shall consider two main types of elimination election procedures in the next three sections. One of these first orders the alternatives to give a voting order $\sigma(\pi)$. It then determines a winner by sequential majority comparisons according to $\sigma(\pi)$, with one alternative eliminated on each comparison. Sequential majority elimination is discussed in Section 5.

The other type also eliminates one (or more) alternatives in each stage but does this solely on the basis of the alternatives that remain at that stage. For a given $\pi\in\Pi$ for X, let π_A for nonempty $A\subseteq X$ be the profile obtained from π by deleting all alternatives not in A. In addition, let H_k for $k=3,\dots,m$ with $|X|=m$ be the set of all real valued functions defined on the Cartesian product of a generic k-element subset of X and the set of all profiles on that subset. We then define a general class F_e of elimination election procedures as follows.

Definition. Given X with m elements, $f\in F_e$ if and only if there are $h_k\in H_k$ for $k=3,\dots,m$ such that $f(\pi)$ is determined as follows for every $\pi\in\Pi$. Begin with $A=X$. Compute $h_k(a,\pi_A)$ for all $a\in A$ when $|A|=k$, and let A^* be the set of all $a\in A$ that *do not* have the smallest value of $h_k(a,\pi_A)$ over A. Then:

if $|A^*|=0$, take $f(\pi)=A$;

if $|A^*|=1$, take $f(\pi)=A^*$;

if $|A^*|=2$ with $A^*=\{a,b\}$, take $f(\pi)=\{a\}$ if $aM(\pi)b$, $f(\pi)=\{b\}$ if $bM(\pi)a$, and $f(\pi)=\{a,b\}$ otherwise;

if $|A^*|\geq3$, repeat the process with the new A equal to the current A^*.

According to the definition, the lowest-scoring alternatives in a given stage are eliminated on the basis of the appropriate h_k applied to the reduced profile for that stage unless all alternatives have the same lowest score ($|A^*| = 0$). If we reach a stage with two alternatives, the elected set is determined by the simple majority relation on those two alternatives.

The next section considers procedures in F_e whose h's are based on score vectors. We shall be primarily concerned with the class $F_e(\Lambda_3)$ of procedures for $X = \{x, y, z\}$, where $f \in F_e(\Lambda_3)$ if and only if $f \in F_e$ and there is a $\lambda \in \Lambda_3$ such that, for all $a \in X$ and $\pi \in \Pi$,

$$h_3(a, \pi) = \sum_{i=1}^{3} \lambda_i \pi(a_i).$$

Section 4 then examines procedures in F_e whose h's are based on majority proportions. When $|X| = 3$, we shall use the set G of all symmetric $[g(\alpha, \beta) = g(\beta, \alpha)]$ and continuous real valued functions g on $[0, 1] \times [0, 1]$ that increase in both arguments and have $g(0,0) = 0$ and $g(1,1) = 1$. We then define $F_e(G)$ for $X = \{x, y, z\}$ by $f \in F_e(G)$ if and only if $f \in F_e$ and there is a $g \in G$ such that, for all $a \in X$ and $\pi \in \Pi$,

$$h_3(a, \pi) = g(\pi(a, b), \pi(a, c))$$

when $\{a, b, c\} = \{x, y, z\}$. As π changes in such a way that a's standing relative to b and c improves, $h_3(a, \pi)$ increases. Procedures in $F_e(G)$ first eliminate the alternative with the worst joint showing against the other two according to a particular g function.

3. Score vector elimination

This section considers successive elimination when score vectors are used to effect eliminations by lowest scores with

$$h_k(a, \pi_A) = \sum_{i=1}^{k} \lambda_i^{(k)} \pi_A(a_i)$$

when $|A| = k$ and $\lambda^{(k)} \in \Lambda_k$. The main result is essentially due to Smith [15].

Our basic conclusion is that all $f \in F_e$ for $m \geq 3$ that are based on elimination by score vectors as described above are nonmonotonic. Since cases with $m > 3$ are analyzed in a manner that is very similar to the $m = 3$ analysis, I shall focus on $F_e(\Lambda_3)$ for the three-alternative case.

We begin with a general result that does not depend explicitly on selection by elimination, but which will be used shortly in connection with $F_e(\Lambda_3)$.

Theorem 1. *Suppose $X = \{x, y, z\}$ and $f \in F$ satisfies the following condition for all $\pi \in \Pi$:*

$$[\{a, b, c\} = \{x, y, z\}, aD(\pi)c, bD(\pi)c, aM(\pi)b] \Rightarrow f(\pi) = \{a\}.$$

Then f is not monotonic.

Remark. The condition in Theorem 1 illustrates the sense of "paradox" intended in this paper. The condition says that if one alternative is positionally inferior to the other two in that it is strongly dominated by each, then the elected set shall be determined (in part) by simple majority between the dominating alternatives. On the face of it, this condition does not seem unreasonable. However, it is incompatible with monotonicity.

Proof of Theorem 1. Let π and π' be the following 27-voter profiles in which π' is obtained from π by moving x ahead of y in three of the four yxz rankings and in the two zyx rankings:

	π	π'
xyz	6	9
zxy	6	8
yzx	6	6

	π	π'
yxz	4	1
zyx	2	0
xzy	3	3 .

It is easily checked that $M(\pi')=M(\pi)$ and that strict majorities cycle, with $xMyMzMx$. Moreover, $\{xD(\pi)z, yD(\pi)z\}$ and $\{xD(\pi')y, zD(\pi')y\}$. The condition of the theorem then gives $f(\pi)=\{x\}$ and $f(\pi')=\{z\}$, and therefore f is not monotonic. □

As x moves ahead of y in several rankings of π to yield π' in the preceding proof, y is moved down enough so that dominance between y and z is reversed, from $yD(\pi)z$ to $zD(\pi')y$. Because of the x-dominances and the majorities, this switch changes the winner from x to z. The third alternative, y, which has the most first-place votes in π and would therefore win under simple plurality voting, plays the role of x's spoiler: by doing x the favor of "transferring" some of its votes to x by the noted inversions, y causes x to lose the election.

Corollary 1. *Every $f \in F_e(\Lambda_3)$ is nonmonotonic.*

Proof. Lemma 1 and the definition of $F_e(\Lambda_3)$ imply that the condition in Theorem 1 holds for every $f \in F_e(\Lambda_3)$. □

The elimination procedures in $F_e(\Lambda_3)$ are often modified by having no runoff between the two alternatives with the largest h_3 values when some alternative gets more than a specified percent of first-place votes. For example, the *single transferable vote method* [6] for $m=3$ determines a winner without elimination if some alternative gets more than 50 percent of the first-place votes. Otherwise, the alternative with the fewest first-place votes is eliminated (i.e., $\lambda=(1,0,0)$), and the winner is determined by majority between the other two. Every such procedure also violates monotonicity so long as the critical percent exceeds 100/3.

Corollary 2. *Given $X=\{x,y,z\}$ and $\delta>0$, let $F_e(\delta)$ be the set of all $f\in F_e(\Lambda_3)$, modified so that if one of the three alternatives has more first-place votes than the others and is ranked first in more than $100(\delta+\frac{1}{3})$ percent of the terms of π, then it is the unique alternative in $f(\pi)$. Every $f\in F_e(\delta)$ is nonmonotonic.*

Proof. We generalize the profiles in the proof of Theorem 1 as follows:

	π	π'
xyz	N	$N+3$
zxy	N	$N+2$
yzx	N	N
yxz	$n+2$	$n-1$
zyx	n	$n-2$
xzy	$n+1$	$n+1$.

As before, three yxz rankings are changed to xyz, and two zyx rankings are changed to zxy in going from π to π'. So long as $n\geq 2$ and $N>n+3$, majorities cycle, with $xMyMzMx$, and $\{xD(\pi)z, yD(\pi)z\}$ and $\{xD(\pi')y, zD(\pi')y\}$. The maximum number of first-place votes for an alternative in either π or π' is $N+n+4$ for x in π'. Given any $\delta>0$, choose n and N so that $n\geq 2$, $N>n+3$ and $(N+n+4)/(3N+3n+3)<\delta+\frac{1}{3}$. Given $f\in F_e(\delta)$, it then follows from Lemma 1 and the majorities that $f(\pi)=\{x\}$ and $f(\pi')=\{z\}$. □

One can in fact go further than Corollary 1 to generate a sequence of monotonicity violations as follows. Consider the three profiles ($N=11, n=7$)

	π	π'	π''
xyz	11	14	14
zxy	11	13	19
yzx	11	11	17
yxz	9	6	0
zyx	7	5	5
xzy	8	8	2

in which π and π' are as in the preceding proof, and π'' is obtained from π' by moving z ahead of x in the six yxz rankings and in six of the eight xzy rankings. Then, for any $f\in F_e(\Lambda_3)$, or for any f that satisfies the condition of Theorem 1, $f(\pi)=\{x\}$, $f(\pi')=\{z\}$, and $f(\pi'')=\{y\}$.

More generally, if K is any positive integer, and if f for $X=\{x,y,z\}$ satisfies the condition of Theorem 1, then there are profiles $\pi_1,\pi_2,\dots,\pi_{K+1}$, each of which beyond the first is obtained from its predecessor by moving its predecessor's elected alternative up in several rankings, such that every $f(\pi_k)$ is a singleton and $f(\pi_{k+1})\neq f(\pi_k)$ for $k=1,\dots,K$. I leave the proof to the reader.

4. Majority proportion elimination

We now consider election procedures in the class $F_e(G)$ for $X=\{x, y, z\}$ defined at the end of section 2, along with two other procedures for $X=\{x, y, z\}$ that are based on elimination by majority proportions but are not in $F_e(G)$ because they have no corresponding g functions that strictly increase in both arguments. The latter two procedures will be denoted f_1 and f_2 and are based, respectively, on $h_3(a,\pi)=\min\{\pi(a,b),\pi(a,c)\}$ and $h_3(a,\pi)=\max\{\pi(a,b),\pi(a,c)\}$ when $\{a,b,c\}=\{x,y,z\}$. Procedure f_1 first eliminates the alternative that is beaten the worst by some other alternative, and f_2 first eliminates the alternative whose best showing against the others is worst.

It turns out that f_2 is the only monotonic election procedure in $F_e(G)\cup\{f_1,f_2\}$. After proving monotonicity for f_2 later in the section, we shall note that its natural extension to $|X|\geq 4$ is not monotonic. In the meantime, our attention will focus on $F_e(G)$.

We note first that every $f\in F_e(G)$ is a Condorcet election procedure. To see this, let $\alpha=\pi(x,y)$, $\beta=\pi(x,z)$ and $\gamma=\pi(y,z)$ so that the h_3 values for the relevant $g\in G$ are

$$h_3(x,\pi)=g(\alpha,\beta),$$
$$h_3(y,\pi)=g(1-\alpha,\gamma),$$
$$h_3(z,\pi)=g(1-\beta,1-\gamma).$$

Suppose $\min\{\alpha,\beta\}>\frac{1}{2}$, so that $xM(\pi)y$ and $xM(\pi)z$. Then

$$g(\alpha,\beta)>\min\{g(1-\alpha,\gamma),g(1-\beta,1-\gamma)\},$$

which implies that $f(\pi)=\{x\}$ for the $f\in F_e(G)$ that is based on g.

The additive g function, defined by

$$g(\alpha,\beta)=\tfrac{1}{2}(\alpha+\beta),$$

corresponds to the $f\in F_e(G)$ that has been referred to as Nanson's function [3,9]. It is equivalent to the Borda-elimination procedure in $F_e(\Lambda_3)$ that uses $\lambda=(2,1,0)$. As noted in [10], the latter procedure is the only one in $F_e(\Lambda_3)$ – up to a positive affine transformation on the components of λ – that is a Condorcet procedure. Therefore $F_e(\Lambda_3)$ and $F_e(G)$ have exactly one procedure in common.

Despite the fact that these two classes of election procedures are almost disjoint, it might be true that procedures in $F_e(G)$ satisfy the condition in Theorem 1. However, this is not the case, as we show with the following 37-voter profile:

	π
xyz	12
zxy	10
yzx	10

	π
yxz	2
xzy	3 .

Clearly, $xM(\pi)yM(\pi)zM(\pi)x$, and, since $(x_1, x_2, x_3) = (15, 12, 10)$, $(y_1, y_2, y_3) = (12, 12, 13)$ and $(z_1, z_2, z_3) = (10, 13, 14)$, we have $xD(\pi)z$ and $yD(\pi)z$. Hence $f(\pi) = \{x\}$ for every f that satisfies the condition in Theorem 1. Suppose, however, that $g \in G$ is such that $g(20/37, 13/37) > g(24/37, 12/37)$. Then, since

$$(\pi(y,z), \pi(y,x)) = (24/37, 12/37), \qquad (\pi(z,x), \pi(z,y)) = (20/37, 13/37),$$

the corresponding $f \in F_e(G)$ will eliminate y in the first stage (x always stays in) and give $f(\pi) = \{z\}$.

Theorem 2. *Every $f \in F_e(G)$ is nonmonotonic.*

Proof. With π and π' specified by

	π	π'
xyz	3	$N+3$
zxy	$N+2$	$N+2$
yzx	2	2
yxz	N	0

π' is obtained from π by moving x ahead of y in the N yxz rankings of π. As in previous examples, we have $M(\pi) = M(\pi')$ with $xMyMzMx$. Let $\mu = [2N+7]^{-1}$. Then, for the G context,

$$h_3(x, \pi) = g((N+5)\mu, (N+3)\mu),$$

$$h_3(y, \pi) = g((N+5)\mu, (N+2)\mu),$$

$$h_3(z, \pi) = g((N+4)\mu, (N+2)\mu),$$

and therefore every $f \in F_e(G)$ will eliminate z in the first stage and give $f(\pi) = \{x\}$ since $xM(\pi)y$. Since

$$h_3(x, \pi') = g((2N+5)\mu, (N+3)\mu),$$

$$h_3(y, \pi') = g((N+5)\mu, 2\mu),$$

$$h_3(z, \pi') = g((N+4)\mu, (N+2)\mu),$$

$f \in F_e(G)$ will eliminate either y or z (possibly both) for π' in the first stage. Now as $N \to \infty$, $h_3(y, \pi') \to g(\frac{1}{2}, 0)$ and $h_3(z, \pi') \to g(\frac{1}{2}, \frac{1}{2})$. Since $g(\frac{1}{2}, \frac{1}{2}) > g(\frac{1}{2}, 0)$, it follows from continuity for g that $h_3(y, \pi') < h_3(z, \pi')$ for sufficiently large N. Hence every $f \in F_e(G)$ will have some N at which y is eliminated for π' in the first stage, thus giving $f(\pi') = \{z\}$. $\square$

We now return to f_1 and f_2 defined at the outset of this section. It is easily seen that both are Condorcet procedures, and a slight modification in the profiles used in the proof of Theorem 2 shows that f_1 is not monotonic. Thus, the following theorem verifies that f_2 is the only elimination procedure considered thus far for $X=\{x, y, z\}$ that is monotonic.

Theorem 3. *f_2 is monotonic.*

Proof. Let π and π' be any two profiles for which π' is obtained from π by moving x up in some rankings. We assume that $f_2(\pi)=\{x\}$, and show that $x\in f_2(\pi')$. Let

$$\alpha=\max\{\pi(x,y),\pi(x,z)\},$$

$$\beta=\max\{\pi(y,x),\pi(y,z)\},$$

$$\gamma=\max\{\pi(z,x),\pi(z,y)\},$$

and let α', β' and γ' be defined similarly for π'. Clearly, $\alpha'\geq\alpha$, $\beta'\leq\beta$ and $\gamma'\leq\gamma$. In addition, let M^* represent majority win or tie, so that $aM^*(\pi)b$ iff either $aM(\pi)b$ or $\pi(a,b)=\pi(b,a)$.

Suppose first that $xM^*(\pi)yM^*(\pi)zM^*(\pi)x$. Then $\beta=\pi(y,z)=\pi'(y,z)=\beta'$, $\alpha=\pi(x,y)$ and $\gamma=\pi(z,x)$. Given $f_2(\pi)=\{x\}$, we require $\alpha>\min\{\beta,\gamma\}$. Moreover, $\beta\geq\gamma$, for otherwise $z\in f_2(\pi)$, and $f_2(\pi)=\{x\}$ further requires $\alpha>\beta$. Hence $\alpha'>\beta'=\beta\geq\gamma\geq\gamma'$, and therefore $f_2(\pi')=\{x\}$. A similar proof shows that monotonicity holds if $xM^*(\pi)zM^*(\pi)yM^*(\pi)x$.

Suppose then that there is no $M^*(\pi)$ cycle and, for definiteness, assume that $yM^*(\pi)z$, so that either $yM(\pi)x$ or $xM(\pi)z$. If both $yM(\pi)x$ and $xM(\pi)z$, then z is eliminated in the first round and $f_2(\pi)=\{y\}$, contrary to $f_2(\pi)=\{x\}$. If $yM(\pi)x$, and $zM^*(\pi)x$, then x will be eliminated in the first round, contrary to $f_2(\pi)=\{x\}$. Therefore, $xM^*(\pi)y$ and $xM(\pi)z$, so $\alpha>\frac{1}{2}\geq\gamma$ and $\beta\geq\frac{1}{2}\geq\gamma$. Then either $\beta=\frac{1}{2}=\gamma$, hence $f_2(\pi)=\{x\}$, or $\beta>\gamma$, in which case $f_2(\pi)=\{x\}$ requires $xM(\pi)y$. Since $\beta'=\beta$, either case will give $f_2(\pi')=\{x\}$. □

The natural extension of f_2 for larger sets eliminates the alternatives with the smallest value of $\max\{\pi(a,b):b\in A\setminus\{a\}\}$ when A is the remainder at any stage, unless all remaining alternatives have the smallest max value. The failure of this extension to be either Condorcet or monotonic is illustrated with the following four-alternative profiles:

	π	π'	
wxyz	8	8	
yzxw	6	6	
zxyw	4	4	
wzxy	3	0	(move x ahead of z for π')
wxzy	0	3 .	

Here w has an 11-to-10 majority over each of the other three alternatives, but $f_2(\pi)=\{x\}$ for the extended f_2: the max $\hat{\pi}$ values for (x, y, z, w) are (15, 14, 13, 11), so

w is eliminated first; the max $\hat{\pi}$ values for (x, y, z) in the reduced profile are $(15, 14, 13)$ so z is eliminated next, and $xM(\pi)y$. Hence the extended f_2 is not Condorcet. The initial max $\hat{\pi}'$ values are the same as the max $\hat{\pi}$ values, except that the value for z drops from 13 to 10. Therefore z is eliminated first under π'. The max $\hat{\pi}'$ values for (x, y, w) in the reduced profile are $(15, 10, 11)$ so that y is eliminated next. Finally, $f_2(\pi') = \{w\}$ since $wM(\pi')x$. Therefore the extended f_2 is not monotonic.

5. Sequential majority elimination

A sequential majority elimination procedure for $|X| = m$ first specifies an order of voting $\sigma(\pi)$ for profile π and then makes a series of $m-1$ pairwise majority comparisons between alternatives according to $\sigma(\pi)$. The initial comparison is between the first two alternatives in $\sigma(\pi)$, and the kth comparison for $k > 1$ is between the $k+1$st alternative in $\sigma(\pi)$ and the winner of the preceding comparison. The alternative that wins the final comparison is elected.

The winner of each comparison is decided by simple majority unless there is a tie. Ties can be broken in several ways [e.g. in favor of the alternative that comes earlier in $\sigma(\pi)$, or in favor of the alternative that comes later in $\sigma(\pi)$], or the procedure can be modified to retain tied alternatives unless they are beaten by something that comes later in $\sigma(\pi)$. In the present discussion, I shall be primarily concerned with results that do not depend on how ties are handled. A related problem with ties can arise in forming $\sigma(\pi)$ when the voting order depends on π, but I shall not dwell on this.

In general, sequential majority elimination procedures are Condorcet. Moreover, if the order of voting is fixed, with $\sigma(\pi) = \sigma(\pi')$ for all $\pi, \pi' \in \Pi$, then these procedures are monotonic for most reasonable ways of handling majority ties. For example, suppose ties are decided in favor of later alternatives in $\sigma(\pi)$. If π' is obtained from π by moving x up in several rankings of π, then x does as well under π' as under π against any other alternative, and comparisons between other alternatives do not change. Therefore, $f(\pi') = \{x\}$ if $f(\pi) = \{x\}$.

However, if $\sigma(\pi)$ changes, then a sequential majority elimination procedure may be nonmonotonic. For example, if $|X| = 3$ and the first alternative in $\sigma(\pi)$ is the one with the lowest point score for some score vector $\lambda \in \Lambda_3$, and if the second alternative in $\sigma(\pi)$ is the majority loser between the other two, then the proof of Theorem 1 shows that the procedure violates monotonicity. On the other hand, if the order of voting is determined on the basis of the λ scores for all three alternatives, without a runoff to determine second place in $\sigma(\pi)$, then monotonicity must hold so long as ties are not involved.

Theorem 4. *Given $X = \{x, y, z\}$ and $\lambda \in \Lambda_3$, suppose σ is determined by lowest to highest point totals under λ for the sequential majority elimination procedure f.*

Suppose further that $\pi, \pi' \in \Pi$, $\sigma(\pi)$ and $\sigma(\pi')$ are unambiguously determined by the λ scores, and there are no majority ties under π or π'. If $f(\pi) = \{x\}$ and π' is obtained from π by moving x up in several rankings of π, then $f(\pi') = \{x\}$.

Proof. Suppose all the hypotheses hold. Then either x has a strict majority over each of y and z under π, in which case the same is true for π', so $f(\pi') = \{x\}$, or there is a majority cycle, say $xM(\pi)yM(\pi)zM(\pi)x$, and the order of voting is either $\sigma(\pi) = yzx$ or $\sigma(\pi) = zyx$. As x moves up in rankings of π to yield π', its λ score does not decrease, and the λ scores of y and z do not increase. Hence either $\sigma(\pi') = yzx$ or $\sigma(\pi') = zyx$ when $M(\pi)$ cycles, and in either case $f(\pi') = \{x\}$ since $yM(\pi')z$ and $xM(\pi')y$. □

As we shall prove shortly, the result of Theorem 4 does not hold when $m \geq 4$ and $\sigma(\pi)$ is determined by low to high scores under a $\lambda \in \Lambda_m$. In fact, we shall prove a stronger result for $m = 4$ (which generalizes easily to $m > 4$) to the effect that a sequential majority elimination procedure must be nonmonotonic so long as a precedes b in $\sigma(\pi)$ if $bD(\pi)a$.

The reader may have noticed that the methods we have mentioned for forming $\sigma(\pi)$ tend to place "weaker" alternatives early in the voting order. Hence, to win the election, a "weaker" alternative must win more pairwise majority contests than an alternative that appears later in the voting order. Beyond this, my main reason for putting "weaker" alternatives early in $\sigma(\pi)$ has to do with *Pareto optimality*, which requires that if every voter ranks a ahead of b in π, then $b \notin f(\pi)$. It is well known that some sequential majority procedures are not Pareto optimal when $m \geq 4$. The simplest example is $\pi = (xyzw, wxyz, zwxy)$ with $\sigma(\pi) = wxzy$, where $f(\pi) = \{y\}$ although all three voters prefer x to y. However, if σ is determined (up to ties) by low to high scores for any strictly monotonic ($\lambda_1 > \lambda_2 > \cdots > \lambda_m$) score vector $\lambda \in \Lambda_m$, then f must be Pareto optimal. For, when such a λ is used, and when every voter in π ranks a ahead of b, then b must precede a in $\sigma(\pi)$, and if b survives until it faces a then it will be eliminated at that time. (If everyone prefers a to b, then the strong dominance relationship $aD(\pi)b$ is not automatic although a weaker form of positional dominance [8, p. 544] does hold.)

Theorem 5. *Suppose $|X| = 4$ and f is a sequential majority elimination procedure for which a precedes b in $\sigma(\pi)$ whenever $bD(\pi)a$. Then f is not monotonic.*

Proof. Let $X = \{x, y, z, w\}$, and consider the following 61-voter profiles π and π', where π' is obtained from π by moving x ahead of y in each of the three-voter rankings under π:

	π	π'
$xyzw$	12	15
$yxzw$	3	0
$wxyz$	7	10
$wyxz$	3	0
$wzxy$	2	5
$wzyx$	3	0

	π	π'
$xwyz$	5	5
$yxwz$	4	4
$yzwx$	11	11
$zwxy$	10	10
$wxzy$	1	1 .

In π, x has majorities over y and z, y has a majority over z, z has a majority over w, and w has majorities over x and y. Since the only changes from π to π' involve x and y, $M(\pi')=M(\pi)$.

The position counts for the four alternatives under π are

	first	*second*	*third*	*fourth*
x	17	15	15	14
y	18	15	15	13
z	10	16	16	19
w	16	15	15	15

Therefore $yD(\pi)xD(\pi)wD(\pi)z$, so $\sigma(\pi)$ is $zwxy$. The noted majorities yield x as the majority elimination winner. Under π', the position counts for z and w do not change, and the counts for x and y are (20, 15, 15, 11) and (15, 15, 15, 16), respectively. Therefore $xD(\pi')wD(\pi')yD(\pi')z$, so $\sigma(\pi')$ is $zywx$. The majorities then give w as the winner. □

6. A nonelimination procedure

Thus far in this paper, the election procedures that have been shown to violate monotonicity all involve some form of successive elimination. We have seen that three very different types of elimination procedures are nonmonotonic. At the same time, some elimination procedures satisfy monotonicity: e.g., f_2 in Section 4, and sequential majority elimination with a fixed order of voting, from Section 5. However, the latter procedures violate Pareto optimality when $m \geq 4$, and the natural extension of f_2 to $m \geq 4$ is neither Condorcet nor monotonic.

In contrast to the plethora of elimination procedures that violate monotonicity, almost all reasonable-sounding election procedures that have been proposed in the literature and which do not involve determination of $f(\pi)$ by successive elimination are monotonic. In order not to give the impression that the latter procedures are all monotonic, we conclude by examining a Condorcet election procedure that is similar to what I have referred to elsewhere [9] as Dodgson's function. The version of this function that is considered below is denoted as f_3.

Procedure f_3 is based on the fewest inversions in a profile's rankings that are needed to produce a Condorcet alternative. Given $x \in X$ and $\pi \in \Pi$, we define $I(x, \pi)$

as the fewest inversions in the rankings of π that are needed to yield a profile π' for which $xM(\pi')y$ for each $y \in X \setminus \{x\}$. As usual, an inversion in a ranking occurs each time that two adjacent alternatives are interchanged. Clearly, $I(x, \pi) = 0$ if and only if $xM(\pi)y$ for all $y \neq x$. We then define f_3 by

$$f_3(\pi) = \{x \in X \colon I(x, \pi) \leq I(y, \pi) \text{ for all } y \in X\}.$$

An example on p. 478 of [9] shows that f_3 is nonmonotonic when $m \geq 5$. The following theorem covers $m \in \{3, 4\}$.

Theorem 6. *f_3 is monotonic when $|X| = 3$ but not when $|X| = 4$.*

Proof. We consider first the proof for $m = 4$ with $X = \{x, y, z, w\}$ since this proof is instructive in seeing how monotonicity can fail for f_3. Let π and π' be the following 43-voter profiles, where π' is obtained from π by moving x ahead of y in two of the five initial rankings of π:

	π	π'
$yxzw$	5	3
$xyzw$	0	2
$ywzx$	9	9

	π	π'
$xzyw$	5	5
$xywz$	9	9
$zxwy$	15	15.

The majority counts for π are

$$\hat{\pi}(x, y) = 29, \qquad \hat{\pi}(x, w) = 34,$$

$$\hat{\pi}(y, z) = 23, \qquad \hat{\pi}(y, w) = 28,$$

$$\hat{\pi}(z, x) = 24, \qquad \hat{\pi}(z, w) = 25.$$

Then, since three zx to xz inversions in the final rankings give x a bare (22 to 21) majority over z, $I(x, \pi) = 3$. Although only two yz to zy inversions are needed to get z over y, each of these entails a second inversion since every ranking of π that has y ahead of z has a third alternative between y and z. Therefore $I(z, \pi) = 4$ and, since $I(y, \pi)$ and $I(w, \pi)$ clearly exceed 3, $f_3(\pi) = \{x\}$.

After the two yx to xy inversions from π to π', two rankings of π' that have y ahead of z have y and z adjacent, and therefore $I(z, \pi') = 2$. As before, $I(x, \pi') = 3$, and it follows that $f_3(\pi') = \{z\}$. This completes the proof for $|X| = 4$.

Assume henceforth that $X = \{x, y, z\}$ and suppose, contrary to the theorem, that f_3 is not monotonic in this case. Then, for definiteness, assume that π' is obtained from π by moving x up, and that

$$I(x, \pi) < \min\{I(y, \pi), I(z, \pi)\},$$

$$I(z, \pi') < I(x, \pi').$$

This can only be true if π has no Condorcet alternative (strict majorities over the others). Moreover, since $I(x, \pi') \leq I(x, \pi)$, it follows that $1 \leq I(z, \pi') < I(x, \pi') \leq$

$I(x,\pi)<I(z,\pi)$, hence that $I(z,\pi')+2\le I(z,\pi)$ and $3\le I(z,\pi)$. Since $I(z,\pi)\ge 3$, something else must have a strict majority over z in π. Moreover, since $1<I(x,\pi)<I(y,\pi)$, $I(y,\pi)\ge 3$ and something else must have a strict majority over y in π. And x cannot do the job in both cases since otherwise it would be the Condorcet alternative for π.

Therefore either $zM(\pi)y$ and $xM(\pi)z$ (hence $yM^*(\pi)x$ also), or $yM(\pi)z$ and $xM(\pi)y$ (hence $zM^*(\pi)x$ also). Now as x moves up in π, it is easily seen that the number of xz to zx inversions in order to get $zM(\pi')x$ cannot be fewer than the number needed to get $zM(\pi)x$. Hence it is impossible to have $I(z,\pi')<I(z,\pi)$ under the initial conditions given above when $zM(\pi)y$ and $xM(\pi)z$. We therefore conclude that if these conditions hold, then $yM(\pi)z, xM(\pi)y$ and $zM^*(\pi)x$, where M^* stands for majority win or tie.

Given the latter majorities, and in view of the comment in the preceding paragraph on xz to zx inversions, the only way to get $I(z,\pi')+2\le I(z,\pi)$ is to force at least two yz to zy inversions to come from rankings yxz of π in going from $yM(\pi)z$ to a profile that has z over y. But for this to be true, the number of yxz rankings of π must exceed the total of all other rankings (i.e. xyz and yzx along with zyx, zxy and xzy), and since this would give y as the Condorcet alternative, we obtain a contradiction. □

Theorem 6 joins earlier results in sections 4 and 5 to show that various generic election procedures are monotonic when $m=3$ but not when $m\ge 4$. This suggests that there may be other reasonable-sounding procedures that only become nonmonotonic when $m>4$. Thus far I know of none.

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